HOMEWORK 2
(Math 250 A, B)

1. (a) Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that: (20 pts)

   (i) $p_A(x) = x^2 - \text{Tr}(A)x + \det(A)$

   (ii) The eigenvalues of A are $\lambda_{1,2} = \dfrac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$.

   (b) Let A be an nxn invertible matrix. Then, show that:

   (i) All the eigenvalues of A are non-zero.

   (ii) The eigenvalues of A^{-1} are of the form $\dfrac{1}{\lambda}$, where λ is an eigenvalue of A.

   (iii) $p_{A^{-1}}(x) = \dfrac{(-x)^n}{\det(A)} p_A(1/x)$.

   (*Hint*: For (b) part (iii) , use the fact that det(AB) = det(A)det(B))

2. (a) If A is an nxn diagonalizable matrix, then show that $A^k = P^{-1}D^kP$, where $k \in \mathrm{N}$, D is diagonal, and P is some invertible matrix . (20 pts)

   (b) Use part (a) to find A^{11}, where $A = \begin{pmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{pmatrix}$.

   (*Hint*: For (a), use the fact that since A is diagonalizable then A is *similar* to a diagonal matrix D)

3. Show that A and B are orthogonal vectors in an inner product space V if and only if

$$\|X\|^2 = \|A\|^2 + \|B\|^2$$

   where $X = A + B$. The above, is usually called the *Generalized Pythagorean Theorem* since it generalizes the classical Pythagorean Theorem in R^2 to any inner product space. Verify Pythagoras's Theorem for the elements $p(x) = x$, $q(x) = x^2$ and $r(x) = x + x^2$ of $P_2[x]$. Are the vectors $p(x)$ and $q(x)$ orthogonal? (30 pts)

   (*Note*: Above, we used the "standard" norm in $P_2[x]$, i.e. $\| p \| = \sqrt{\int_{-1}^{1} [p(x)]^2 \, dx}$)

4. Consider the following three norms on $\mathbb{R}^2$: (30 pts)

$$\|\vec{u}\|_\infty = \max\{|u_i|\} \ , \quad \|\vec{u}\|_1 = |u_1| + |u_2| \ , \quad \|\vec{u}\|_2 = \sqrt{u_1^{\,2} + u_2^{\,2}}$$

The above norms are called the *infinity-norm* (or max-norm), *one-norm* (or taxicab norm), and *two-norm* (or Euclidean norm) respectively. Notice that the two-norm, our standard norm in $\mathbb{R}^2$, is the norm induced by the standard inner product in $\mathbb{R}^2$.

(a) Given $\vec{u} = (1,-2)$, compute the length of $\vec{u}$ with respect to each of the three norms above.

(b) Describe how a unit circle (or a 1-sphere) in $\mathbb{R}^2$ will look like with respect to each of the three norms above. Give the equation of the circle in each case, and plot its graph.

(c) Can you describe the unit *hyperspheres* in $M_{2\times 2}(\mathbb{R})$ and $P_2[x]$ respectively, (i.e. the 3-sphere, and 2-sphere, respectively), where the norms for each space are the standard norms of $M_{2\times 2}(\mathbb{R})$ and $P_2[x]$?

(*Note*: An *n*-sphere is a sphere of dimension *n*. For example, the ordinary unit sphere in $\mathbb{R}^3$, i.e. $x^2 + y^2 + z^2 = 1$, is a 2-sphere)