Spring '07

Math 140 Test 4 Solutions

1. Given $A = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -3 & 2 \\ 3 & -9 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, examine which of the two is invertible without calculating any inverses . (10 pts)

Square matricies are invertable $\Longleftrightarrow |M| \neq 0$

$$|A| = \left| \begin{matrix} -1 & 2 & 0 \\ 1 & -3 & 2 \\ 3 & -9 & 6 \end{matrix} \right| \begin{matrix} -1 & 2 \\ 1 & -3 \\ 3 & -9 \end{matrix} = (-1)(-3)(6) + (2)(2)(3) + (0) - (2)(1)(6) - (-1)(2)(-9) - (0)$$

$$= 18 + 12 - 12 - 18 = 0 \Rightarrow A \text{ is } \underline{\text{NOT}} \text{ invertable.}$$

$$|B| = \left| \begin{matrix} 1 & 3 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{matrix} \right| \begin{matrix} 1 & 3 \\ 0 & -2 \\ 0 & 0 \end{matrix} = (1)(-2)(1) + 0 + 0 - 0 - 0 - 0$$

$$= -2 \neq 0 \Rightarrow B \text{ is invertable}$$

2. True or False: $|A+B| = |A| + |B|$. If your answer is False, make sure you provide a counterexample. (10 pts)
(*Note*: You can use 2x2 matrices for a counterexample)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$|A+B| = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right| = \left| \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \right|$$

$$= (a+e)(d+h) - (c+g)(b+f) = ad + ah + ed + eh - cb - cf - gb - gf$$

$|A| = ad - bc$ $\quad |B| = eh - fg$

$\Rightarrow |A| + |B| = ad + eh - bc - fg$

Thus they are not equal!

3. Consider the triangle with vertices $A = (1,2,0)$, $B = (0,0,0)$, and $C = (2,0,0)$.

(a) Find the 4×4 matrix that first rotates the triangle ABC by $180°$ about the y-axis, and then translates it by $(0,1,0)$.

$$R_y(\theta) = \begin{bmatrix} \cos 180° & 0 & \sin 180° & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 180° & 0 & \cos 180° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

; The matrix that rotates the triangle about the origin and then translates is $T \cdot R$

$$T \cdot R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Use the above matrix to find the coordinates of the shifted triangle.

$$A' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

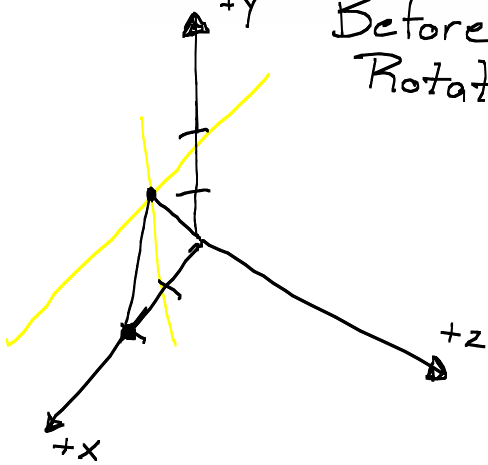
$$C' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

So the new triangle has coordinates:

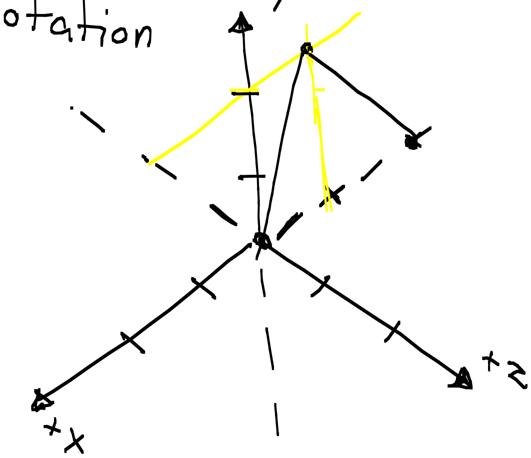
$$\left\{ \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(c) Make all relevant graphs that show the original triangle, the rotation, and the shifted triangle. (30 pts)

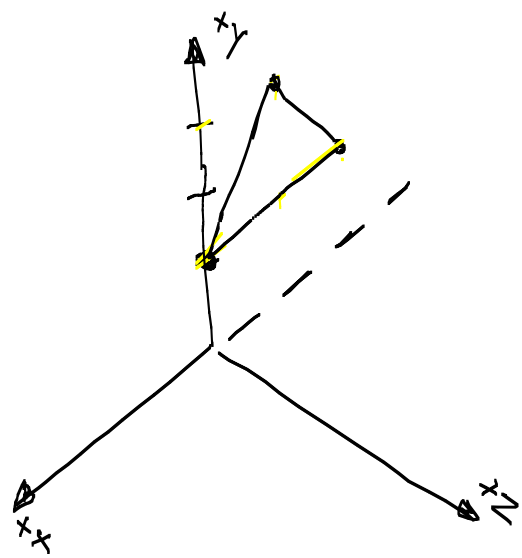
Before Rotation



After Rotation



After Translation



4. Given the basis $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, examine whether it is an orthonormal basis. If your answer is **no**, then use the *Gram-Schmidt Orthogonalization Process* to construct an orthonormal basis from the basis above. (30 pts)
(*Note*: Take the inner product to be the usual **dot** product)

We do not have an orthonormal basis $\because \vec{u} \cdot \vec{w} \neq 0$,
Thus we must construct one

$\vec{u}\,' = \vec{u}$ $\quad \vec{v}\,' = \vec{v} - \dfrac{\vec{v} \cdot \vec{u}\,'}{\vec{u}\,' \cdot \vec{u}\,'} \vec{u}\,' = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} - 0 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$$\vec{w}\,' = \vec{w} - \frac{\vec{w} \cdot \vec{u}\,'}{\vec{u}\,' \cdot \vec{u}\,'} \vec{u}\,' - \frac{\vec{w} \cdot \vec{v}\,'}{\vec{v}\,' \cdot \vec{v}\,'} \vec{v}\,'$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{0}{1} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

5. (a) True or False:

T. (i) A Bezier curve is linear if and only if all control points lie on the curve.
F (ii) Bezier curves are not preserved under affine transformations.

(b) Find the point on the Bezier curve determined by $A = (-1,2)$, $B = (1,2)$, and $C = (4,0)$ that corresponds to $t = 1/4$. (20 pts)

Let $L(u,v,t) = (1-t)u + tv$

The point on the Bezier Curve for $t=a$ is:

$$L(L(A,B,t), L(B,C,t), t)$$

$$L(A,B,\tfrac{1}{4}) = (1-\tfrac{1}{4})A + \tfrac{1}{4}B = \tfrac{3}{4}(-1,2) + \tfrac{1}{4}(1,2) = (-\tfrac{3}{4}, \tfrac{6}{4}) + (\tfrac{1}{4}, \tfrac{2}{4}) = (-\tfrac{3}{4}+\tfrac{1}{4}, \tfrac{6}{4}+\tfrac{2}{4}) = (-\tfrac{1}{2}, 2)$$

$$L(B,C,\tfrac{1}{4}) = (1-\tfrac{1}{4})(1,2) + \tfrac{1}{4}(4,0) = \tfrac{3}{4}(1,2) + (1,0) = (\tfrac{3}{4}, \tfrac{6}{4}) + (1,0) = (\tfrac{7}{4}, \tfrac{3}{2})$$

So $L(L(A,B,t), L(B,C,t), t) = L((-\tfrac{1}{2}, 2), (\tfrac{7}{4}, \tfrac{3}{2}), \tfrac{1}{4})$

$$= \tfrac{3}{4}(-\tfrac{1}{2}, 2) + \tfrac{1}{4}(\tfrac{7}{4}, \tfrac{3}{2}) = (-\tfrac{3}{8}, \tfrac{3}{2}) + (\tfrac{7}{16}, \tfrac{3}{8}) = (\tfrac{1}{16}, \tfrac{15}{8})$$