Math 140 Homework 2 Solutions

Spring '07

1. Show that if A and B are rotations of $\Re^2$, then $AB = BA$. Is the same true for rotations in $\Re^3$? Explain. (20 pts)

We start with two rotation matricies

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad B = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$AB = \begin{bmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha & \cos\theta\sin\alpha + \sin\theta\cos\alpha \\ -\sin\theta\cos\alpha - \cos\theta\sin\alpha & -\sin\theta\sin\alpha + \cos\theta\cos\alpha \end{bmatrix}$$

$$BA = \begin{bmatrix} \cos\alpha\cos\theta - \sin\theta\sin\alpha & \cos\alpha\sin\theta + \sin\alpha\cos\theta \\ -\sin\alpha\cos\theta - \sin\theta\cos\alpha & -\sin\alpha\sin\theta + \cos\theta\cos\alpha \end{bmatrix}$$

The same is not true for $\mathbb{R}^3$. There are many counterex. Consider the following:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus $AB \neq BA$ in $\mathbb{R}^3$.

2. Given $A = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{pmatrix}$, find A^{-1}. (20 pts)

To find the inverse we will find the adjugate matrix. $A^{-1} = \frac{1}{\det|A|} \operatorname{adj}(A)$. This can be thought of as finding the determinate at each position and multi. each component in the cofactor matrix $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$a_{1,1} = 5 - 1 = 4$

$a_{1,2} = -(5 + +2) = -7$

$a_{1,3} = -1 - 2 = -3$

$a_{2,1} = -(5 - 1) = -4$

$a_{2,2} = 20 - -2 = 22$

$a_{2,3} = (-4 - 2)(-1) = 6$

$a_{3,1} = -1 + +1 = 0$

$a_{3,2} = (-1)(-4 + +1) = 3$

$a_{3,3} = 4 - 1 = 3$

$$|A| = \begin{vmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{vmatrix} \begin{matrix} 4 & 1 \\ 1 & 1 \\ 2 & -1 \end{matrix}$$

$$= (20 + -2 + 1) - (5 + 4 - 2)$$

$$= 19 - 7 = 12$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{12} \begin{bmatrix} 4 & -7 & -3 \\ -4 & 22 & 6 \\ 0 & 3 & 3 \end{bmatrix}^T$$

$$= \frac{1}{12} \begin{bmatrix} 4 & -4 & 0 \\ -7 & 22 & 3 \\ -3 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 & 0 \\ -7/12 & 11/6 & 1/4 \\ -1/4 & 1/2 & 1/4 \end{bmatrix}$$

3. Let A be a 3×3 matrix. Show that $|A^t| = |A|$. (20 pts)

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a & d & h \\ b & e & i \\ c & f & j \end{bmatrix}$$

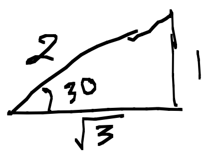
$$|A| = \left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ h & i & j & h & i \end{array}\right| = aej + bfh + cdi - bdj - afi - ceh$$

$$|A^T| = \left|\begin{array}{ccc|cc} a & d & h & a & d \\ b & e & i & b & e \\ c & f & j & c & f \end{array}\right| = aej + dic + hbf - dbj - aif - hec$$

Thus $|A| = |A^T|$

4. Rotation of a figure about a point P in $\Re^2$ is accomplished by first translating the figure by $-P$, rotating about the origin, and then translating back to P. Construct a 3×3 matrix A that rotates points by $30°$ about the point (1,-1), using homogeneous coordinates. (20 pts)

Rotation : $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$



If $\theta = \pi/6$; $\sin\theta = \frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2}$

Since rotation are defined about the origin, we must translate to the origin, rotate, then translate back. All the matricies can be multiplied together to make a single matrix that does all three operations.

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$; $R = \begin{bmatrix} \sqrt{3}/2 & -\frac{1}{2} & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^{-1}RA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 & 1 \\ 1/2 & \sqrt{3}/2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & \frac{-\sqrt{3}-1}{2}+1 \\ 1/2 & \sqrt{3}/2 & \frac{\sqrt{3}-1}{2}-1 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Given the basis $A=\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $B=\begin{pmatrix}0&1\\0&1\end{pmatrix}$, $C=\begin{pmatrix}1&0\\1&0\end{pmatrix}$, $D=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, examine whether it is an orthonormal basis. If your answer is **no**, then use the *Gram-Schmidt Orthogonalization Process* to construct an orthonormal basis out of the one above.

(20 pts)

The basis above is not orthonormal $\therefore$

$A \cdot B = 1 \neq 0$, so...

$$A' = A \qquad B' = B - \frac{B \cdot A'}{A' \cdot A'} = \begin{bmatrix}0&1\\0&1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1&1\\0&0\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}&\frac{1}{2}\\0&1\end{bmatrix}$$

$$C' = C - \frac{C \cdot A'}{A' \cdot A'}A' - \frac{C \cdot B'}{B' \cdot B'}B'$$

$$= \begin{pmatrix}1&0\\1&0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1&1\\0&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-\frac{1}{2}&\frac{1}{2}\\0&1\end{pmatrix} = \begin{pmatrix}\frac{1}{4}&-\frac{3}{4}\\1&-1\end{pmatrix}$$

$$D' = D - \frac{D \cdot A'}{A' \cdot A'}A' - \frac{D \cdot B'}{B' \cdot B'}B' - \frac{D \cdot C'}{C' \cdot C'}C'$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1&1\\0&0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-\frac{1}{2}&\frac{1}{2}\\0&1\end{pmatrix} - \frac{-\frac{3}{4}}{\frac{21}{8}}\begin{pmatrix}\frac{1}{4}&-\frac{3}{4}\\1&-1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{3}{4}+\frac{1}{14} & -\frac{3}{4}-\frac{3}{14}\\ \frac{2}{7} & \frac{1}{2}-\frac{2}{7}\end{pmatrix} = \begin{pmatrix}\frac{23}{28} & -\frac{27}{28}\\ \frac{2}{7} & \frac{3}{14}\end{pmatrix}$$

Orthonomal Basis $= \left\{ \begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}-\frac{1}{2}&\frac{1}{2}\\0&1\end{pmatrix}, \begin{pmatrix}\frac{1}{4}&-\frac{3}{4}\\1&-1\end{pmatrix}, \begin{pmatrix}\frac{23}{28}&-\frac{27}{28}\\\frac{2}{7}&\frac{3}{14}\end{pmatrix} \right\}$