Key

TEST 2

(Math 140)

1. Show that the vectors $\vec{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ are linearly independent. (10 pts)

Let $\lambda_1 \vec{u} + \lambda_2 \vec{v} = \vec{0} \Rightarrow \lambda_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2\lambda_1 - \lambda_2 \\ -\lambda_1 + 4\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \left.\begin{matrix} 2\lambda_1 - \lambda_2 = 0 \\ -\lambda_1 + 4\lambda_2 = 0 \end{matrix}\right\} \Rightarrow \lambda_1 = 0, \ \lambda_2 = 0.$

ie, $\vec{u}, \vec{v}$ lin. indep. //

2. Show that $M_{2\times 2}(R)$ is spanned by the following vectors: (10 pts)

$$\vec{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \vec{E}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \vec{E}_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \vec{E}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Let $A = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_3 + \lambda_4 & \lambda_2 + \lambda_3 \\ \lambda_2 & \lambda_1 + \lambda_3 \end{pmatrix} \Rightarrow \left.\begin{matrix} a = \lambda_1 + \lambda_3 + \lambda_4 \\ b = \lambda_2 + \lambda_3 \\ c = \lambda_2 \\ d = \lambda_1 + \lambda_3 \end{matrix}\right\} \Rightarrow \begin{matrix} \lambda_2 = c, \ \lambda_3 = b - c \\ \lambda_1 = d - b + c \\ \lambda_4 = a - d \end{matrix}$

ie, $A = (d-b+c)E_1 + cE_2 + (b-c)E_3 + (a-d)E_4$

So, E_1, E_2, E_3, E_4 span $M_{2\times2}(\mathbb{R})$ //

3. Given $A = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$, find $AB + B^2 - 2I$. (10 pts)

$$AB + B^2 - 2I = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 3 \\ 2 & -6 \end{pmatrix}$$

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4. Given $A = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$, find A^{-1}. (10 pts)

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{-2} \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3/2 \\ 0 & 1/2 \end{pmatrix}$$

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5. (a) Show that the following map is a <u>linear</u> map: (30 pts)

$$T: R^2 \to R^2$$

$$T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 \\ -u_2 \end{pmatrix}$$

- $T\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right] = T\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix} = \begin{pmatrix} u_1+v_1+u_2+v_2 \\ -(u_2+v_2) \end{pmatrix} = \begin{pmatrix} u_1+u_2+v_1+v_2 \\ -u_2-v_2 \end{pmatrix}$
$= \begin{pmatrix} u_1+u_2 \\ -u_2 \end{pmatrix} + \begin{pmatrix} v_1+v_2 \\ -v_2 \end{pmatrix}$
$= T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

- $T\left[\lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right] = T\begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix} = \begin{pmatrix} \lambda u_1 + \lambda u_2 \\ -\lambda u_2 \end{pmatrix}$
$= \lambda\begin{pmatrix} u_1+u_2 \\ -u_2 \end{pmatrix}$
$= \lambda T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

ie, T is linear. //

(b) Find Ker(T) .

$$\begin{aligned}\mathrm{Ker}(T) &= \left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2 \mid T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \\ &= \left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2 \mid \begin{pmatrix} u_1+u_2 \\ -u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \\ &= \left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2 \mid \begin{matrix} u_1+u_2=0 \\ -u_2=0 \end{matrix}\right\} = \left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2 \mid u_1=0, u_2=0\right\} = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}\end{aligned}$$

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6. Identify the <u>type</u> of linear map the following represent: (20 pts)
(*Note*: By type, we mean whether is a reflection, projection or some other kind of linear map)

(a) $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (b) $B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (c) $C = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$ (d) $D = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

(a) scaling by 2
(b) reflection about the line $y=-x$
(c) Rotation by $60°$
(d) projection about the line $y=-x$.

7. Find the matrix that changes the basis $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ into $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(*Note*: Do <u>not</u> solve any system to find the relevant coefficients. They are easy to find) (10 pts)

$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$.

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