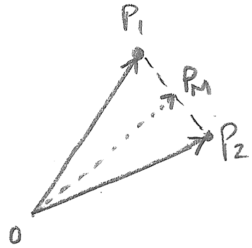
Key

TEST 1

(Math 250 B)

1. (a) Show that the midpoint of $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by: (20 pts)

$$P_M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



(*Hint*: Consider any of the triangles generated by the position vectors OP_1, OP_2 and OP_M)

Let $\vec{OP_1}$, $\vec{OP_2}$ and $\vec{OP_M}$ be the pos. vectors of P_1, P_2 and P_M.

Want to find $\vec{OP_M} = (x, y)$ … (ie, $x, y = ?$)

We have $\vec{P_1P_M} = \frac{1}{2}\vec{P_1P_2}$, bec. P_M is the midpoint.

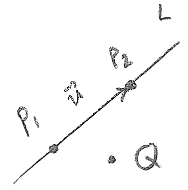
$$\Rightarrow \vec{OP_M} = \vec{OP_1} + \vec{P_1P_M} = \vec{OP_1} + \tfrac{1}{2}\vec{P_1P_2} = \vec{OP_1} + \tfrac{1}{2}(\vec{OP_2} - \vec{OP_1}) = \tfrac{1}{2}(\vec{OP_1} + \vec{OP_2})$$
$$= \tfrac{1}{2}(x_1 + x_2, y_1 + y_2)$$

ie $P_M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ //

(b) Use the above formula to find the midpoint of $P_1 = (1, -1)$ and $P_2 = (\frac{1}{2}, 3)$.

$$P_M = \left(\frac{1+\frac{1}{2}}{2}, \frac{-1+3}{2}\right) = \left(\frac{3/2}{2}, \frac{2}{2}\right) = (3/4, 1).$$ //

2. (a) Find the equation of the line L through the points $P_1 = (-1, 1, 2)$ and $P_2 = (0, 2, 1)$. Does the point $Q = (-1, 1, -1)$ lie on L? (20 pts)

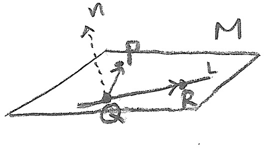


$\vec{u} = \vec{P_1P_2} = \vec{OP_2} - \vec{OP_1} = (1, 1, -1)$ is parallel to L.

So, $L: \begin{cases} x = -1 + t \\ y = 1 + t \\ z = 2 - t \end{cases}$

$Q \notin L$, bec. $\left.\begin{array}{l} \left.\begin{array}{l} -1 = -1 + t \\ 1 = 1 + t \end{array}\right\} \rightarrow t = 0 \\ -1 = 2 - t \longrightarrow t = 3 \end{array}\right\} \neq$, ie $\nexists\, t$ st $L(t) = Q$. //

(b) Find the equation of the plane M that contains the point $P = (1, -1, 2)$ and the line $L: x = t, y = t + 1, z = -3 + 2t$.



A point Q on L is : For $t=0 \rightarrow Q = (0, 1, -3)$

Another is R : For $t=1 \rightarrow R = (1, 2, -1)$

Now, $\vec{QP} = \vec{OP} - \vec{OQ} = (1, -1, 5)$, $\vec{QR} = \vec{OR} - \vec{OQ} = (1, 1, 2)$ $\Rightarrow \vec{n} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ 1 & 2 & 5 \\ 1 & 1 & 2 \end{vmatrix} = \ldots = (-1, 3, -1)$

So, M is : $(-1)(x-1) + (3)(y+1) + (-1)(z-2) = 0 \Rightarrow x - 3y + z = 6$ //

3. The set $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$, equipped with the following *addition* and *scalar multiplication*: (20 pts)

$$x + y = xy$$
$$\lambda \cdot x = x^{\lambda}$$

becomes a vector space. In this vector space, show that:

(a) $1 + 2 = 2$ (b) $0 \cdot 2 = 1$

$1 + 2 = (1)(2) = 2$

$0 \cdot 2 = 2^0 = 1$

//

4. Let $W = \{p(x) \in P_2[x] \mid p(5) = 0\}$. Show that W is a subspace of $P_2[x]$. (10 pts)

(*Note*: W is the set of all quadratic polynomials that have 5 as a root)

Let $p(x), q(x) \in W \Rightarrow p(5) = 0$ and $q(5) = 0$.

Then, $(p+q)(5) = p(5) + q(5) = 0 + 0 = 0 \Rightarrow (p+q)(x) \in W$.

Also, $(\lambda p)(5) = \lambda p(5) = \lambda \cdot 0 = 0 \Rightarrow (\lambda p)(x) \in W$

ie, $W \underset{\text{Subsp.}}{\subset} P_2[x]$.

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5. Let $W = \{A \in M_{2\times2}(\mathbb{R}) \mid \det(A) = 0\}$. Is W a subspace of $M_{2\times2}(\mathbb{R})$? (10 pts)

If your answer is *no*, provide a counterexample.

(*Note*: W is the set of all non-invertible matrices)

W is <u>not</u> a subspace of $M_{2\times2}(\mathbb{R})$, bec. is not closed under $+$.

Indeed, take $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in W$.

Then, $A_1 + A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin W$ (bec. $\det(A_1 + A_2) = 1 \neq 0$)

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6. Show that the vectors $x^2, x+1, 1-x-x^3, 2$ span the vector space $P_3[x]$. (10 pts)

Let $\lambda_1 x^2 + \lambda_2(x+1) + \lambda_3(1-x-x^3) + \lambda_4(2) = ax^3 + bx^2 + cx + d$

$\Rightarrow -\lambda_3 x^3 + \lambda_1 x^2 + (\lambda_2 - \lambda_3)x + \lambda_2 + 2\lambda_4 = ax^3 + bx^2 + cx + d$

$$\Rightarrow \left.\begin{array}{r} -\lambda_3 = a \\ \lambda_1 = b \\ \lambda_2 - \lambda_3 = c \\ \lambda_2 + \lambda_3 + 2\lambda_4 = d \end{array}\right\} \Rightarrow \begin{array}{l} \lambda_1 = b,\ \lambda_2 = c - a \\ \lambda_3 = -a,\ \lambda_4 = \dfrac{d - c + 2a}{2} \end{array}$$

, ie $\exists\ \lambda_1, \lambda_2, \lambda_3, \lambda_4$ st any vector in $P_3[x]$ is a lin. comb. of the vectors $x^2, x+1, 1-x-x^3$ and 2.

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7. Show that the vectors $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ are linearly independent in $M_{2\times2}(\mathbb{R})$. (10 pts)

Let $\lambda_1 A_1 + \lambda_2 A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

$\Rightarrow \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \lambda_1 + \lambda_2 & 0 \\ \lambda_2 & -\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \left.\begin{array}{r} \lambda_1 + \lambda_2 = 0 \\ \lambda_2 = 0 \\ -\lambda_1 = 0 \end{array}\right\} \Rightarrow \lambda_1 = \lambda_2 = 0.$$

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