Key

TEST 2

(Math 250 A)

1. (a) Show that the following map is a <u>linear</u> map: (20 pts)

$$T: R^3 \to R$$
$$T(x,y,z) = 2x - 3y + 4z$$

- $T[(x,y,z) + (x',y',z')] = T(x+x', y+y', z+z') = 2(x+x') - 3(y+y') + 4(z+z')$
$= 2x - 3y + 4z + 2x' - 3y' + 4z'$
$= T(x,y,z) + T(x',y',z')$

- $T[\lambda(x,y,z)] = T(\lambda x, \lambda y, \lambda z)$
$= 2(\lambda x) - 3(\lambda y) + 4(\lambda z)$
$= \lambda(2x - 3y + 4z) = \lambda T(x,y,z)$

ie, T is linear. //

(b) Is the following map linear?

$$T: R^2 \to R^2$$
$$T(x,y) = (x^2, y^2)$$

No, bec. $T[\lambda(x,y)] \neq \lambda T(x,y)$.

Indeed, take $\lambda = 2$ and $(x,y) = (1,-1)$.

Then, $T[2(1,-1)] = T(2,-2) = (4,4) \neq (2,2) = 2(1,1) = 2T(1,-1)$. //

2. (a) Explain why there is a unique linear map $T: R^2 \to R^2$ for which $T(1,-2) = (2,3)$ and $T(3,1) = (0,1)$. (30 pts)

Since $(1,-2)$ and $(3,1)$ are lin. indep. $\Rightarrow$ they are a basis for $\mathbb{R}^2$.
But now that $(1,-2)$, $(3,1)$ are a basis, by a THM on Linear Maps we know that we can construct a unique linear map def. by:
$T(x,y) = \lambda_1(2,3) + \lambda_2(0,1)$
where $(x,y) = \lambda_1(1,-2) + \lambda_2(3,1)$. //

(b) Find a formula for T above.

Let $(x,y) = \lambda_1(1,-2) + \lambda_2(3,1) \Rightarrow \begin{cases} x = \lambda_1 + 3\lambda_2 \\ y = -2\lambda_1 + \lambda_2 \end{cases} \Rightarrow \begin{array}{l} \lambda_1 = \frac{x-3y}{7} \\ \lambda_2 = \frac{2x+y}{7} \end{array}$

So, $(x,y) = \frac{x-3y}{7}(1,-2) + \frac{2x+y}{7}(3,1)$

Hence, $T(x,y) = T\left[\frac{x-3y}{7}(1,-2) + \frac{2x+y}{7}(3,1)\right] = \frac{x-3y}{7}T(1,-2) + \frac{2x+y}{7}T(3,1)$
$= \frac{x-3y}{7}(2,3) + \frac{2x+y}{7}(0,1) = \left(\frac{2x-6y}{7}, \frac{5x-8y}{7}\right)$

ie, $T(x,y) = \left(\frac{2x-6y}{7}, \frac{5x-8y}{7}\right)$ //

(c) Is there a linear map $T: R^2 \to R^2$ for which $T(2,2) = (8,-6)$ and $T(5,5) = (3,-2)$?
(*Hint*: Notice that $(5,5) = 5/2(2,2)$ and explain why this causes a problem)

No. Notice that $(2,2)$ and $(5,5)$ are lin. dep. Indeed, $(5,5) = 5/2\,(2,2)$. (*)
Supp. that $\exists$ T linear with the mentioned properties.
Then, $T(5,5) \overset{(*)}{=} T(5/2(2,2)) \overset{\text{linear}}{=} 5/2\,T(2,2)$
But this is a $\downarrow$ bec $\left.\begin{array}{l} T(5,5) = (3,-2) \\ 5/2\,T(2,2) = 5/2(8,-6) = (20,-15) \end{array}\right\} \neq$ //

3. Consider the linear map defined by: (20 pts)

$$T: M_{2\times 2}(R) \to M_{2\times 2}(R)$$
$$T(A) = A - A^t$$

Show that: (a) Ker(T) = Symm(R)
(b) Im(T) = AntiSymm(R)

(a) Let $A \in \ker(T) \iff \ker(T) = 0 \iff A - A^t = 0 \iff A = A^t \iff A \in \text{Symm}(\mathbb{R})$.
ie, $\ker(T) = \text{Symm}(\mathbb{R})$. //

(b) Let $B \in \text{Im}(T) \Rightarrow B = T(A)$, some $A \in M_{2\times 2}(\mathbb{R})$.
But $B^t = T(A)^t = (A - A^t)^t = A^t - A = -(A - A^t) = -T(A) = -B$
ie, $B^t = -B$
ie, $B \in \text{AntiSymm}(\mathbb{R})$.
ie, $\text{Im}(T) \subset \text{AntiSymm}(\mathbb{R})$ (1)
Conversely, let $B \in \text{AntiSymm}(\mathbb{R}) \Rightarrow B^t = -B$ (*)
But then, for B $\exists$ $A = \tfrac{1}{2}B$ st $B = \tfrac{1}{2}B + \tfrac{1}{2}B \overset{(*)}{=} \tfrac{1}{2}B - (\tfrac{1}{2}B)^t = T(\tfrac{1}{2}B) = T(A)$.
ie, $B \in \text{Im}(T)$
ie, $\text{AntiSymm}(\mathbb{R}) \subset \text{Im}(T)$ (2)
By (1) and (2) we have $\text{Im}(T) = \text{AntiSymm}(\mathbb{R})$ //

4. Let S be an invertible $n \times n$ matrix and consider the linear map defined by:

$$T: M_{n\times n}(R) \to M_{n\times n}(R)$$
$$T(A) = AS$$

Show that T is one-to-one and onto. (20 pts)

one-to-one: Let $T(A) = T(B) \Rightarrow AS = BS \overset{S \text{ invertible}}{\Longrightarrow} (AS)S^{-1} = (BS)S^{-1}$
$\Rightarrow A(SS^{-1}) = B(SS^{-1})$
$\Rightarrow AI = BI$
$\Rightarrow A = B$, ie T one-to-one //

(or Show that $\ker(T) = \{0\}$.
Indeed, let $A \in \ker(T) \Rightarrow T(A) = 0$
$\Rightarrow AS = 0$
$\overset{S \text{ inv.}}{\Longrightarrow} (AS)S^{-1} = 0$
$\Rightarrow A = 0$
$\Rightarrow \ker(T) = \{0\}$)

onto: For any $B \in M_{n\times n}(\mathbb{R})$ $\exists$ $A = BS^{-1} \in M_{n\times n}(\mathbb{R})$ st $B = T(A)$.
Indeed, $T(A) = T(BS^{-1}) = (BS^{-1})S = B(SS^{-1}) = BI = B$.
ie, T onto. //

5. Consider the linear map ("differentiation") defined by: (10 pts)

$$D: P_n[x] \to P_{n-1}[x]$$
$$D(p(x)) = p'(x)$$

Find Ker(D) and use that to conclude that D is onto.
(*Hint*: Use the *Dimension Formula* for the 2nd half of the problem)

Let $p(x) \in \ker(D) \Rightarrow D(p(x)) = 0 \Rightarrow p'(x) = 0 \Rightarrow p(x) = c$, $c \in \mathbb{R}$.
ie, $\ker(D) = \{p(x) \in P_n[x] \mid p(x) = c\} = \{c \mid c \in \mathbb{R}\}$.
That means, $\dim(\ker(D)) = 1$.
From the Dimension Formula, we have: $\dim P_n[x] = \dim(\ker(D)) + \dim(\text{Im}(D))$
$\Rightarrow n+1 = 1 + \dim(\text{Im}(D))$
$\Rightarrow \dim(\text{Im}(D)) = n$.
ie, $\dim(\text{Im}(D)) = \dim P_{n-1}[x]$
$\Rightarrow \text{Im}(D) = P_{n-1}[x]$, ie D onto. //