

TEST 2

(Math 250 B)

1. (a) Show that the following map is a
- linear
- map:

(20 pts)

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$T(x, y) = (x + y, x)$$

$$\begin{aligned} - T[(x, y) + (x', y')] &= T(x + x', y + y') = (x + x' + y + y', x + x') = (x + y + x' + y', x + x') \\ &= (x + y, x) + (x' + y', x') \\ - T[\lambda(x, y)] &= T(\lambda x, \lambda y) = (\lambda x + \lambda y, \lambda x) \\ &= \lambda(x + y, x) \\ &= \lambda T(x, y) \end{aligned}$$

ie, T is linear.

- (b) Is the following map linear?

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$T(x, y) = xy$$

No, bec. $T[(x, y) + (x', y')] \neq T(x, y) + T(x', y')$.Indeed, take $(x, y) = (1, 2)$ and $(x', y') = (2, 1)$.

$$\text{Then, } T[(1, 2) + (2, 1)] = T(3, 3) = 3 \cdot 3 = 9 \neq 4 = 1 \cdot 2 + 2 \cdot 1 = T(1, 2) + T(2, 1).$$

2. (a) Explain why there is a unique linear map
- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$
- for which
- $T(3, 1) = (2, -4)$
- and
- $T(1, 1) = (0, 2)$
- . (30 pts)

Since $(3, 1)$ and $(1, 1)$ are lin. indep. \Rightarrow they are a basis for \mathbb{R}^2 .But now that $(3, 1), (1, 1)$ are a basis, by a THM on Linear Maps we know that we can construct a unique linear map def. by:

- (b) Find a formula for
- T
- above.

$$T(x, y) = \lambda_1(2, -4) + \lambda_2(0, 2)$$

$$\text{where } (x, y) = \lambda_1(3, 1) + \lambda_2(1, 1).$$

$$\text{Let } (x, y) = \lambda_1(3, 1) + \lambda_2(1, 1) \Rightarrow \begin{cases} x = 3\lambda_1 + \lambda_2 \\ y = \lambda_1 + \lambda_2 \end{cases} \Rightarrow \begin{cases} \lambda_1 = \frac{x-y}{2} \\ \lambda_2 = \frac{3y-x}{2} \end{cases}$$

$$\text{So, } (x, y) = \frac{x-y}{2}(3, 1) + \frac{3y-x}{2}(1, 1)$$

$$\text{Hence, } T(x, y) = T\left[\frac{x-y}{2}(3, 1) + \frac{3y-x}{2}(1, 1)\right] = \frac{x-y}{2}T(3, 1) + \frac{3y-x}{2}T(1, 1)$$

$$\text{ie, } T(x, y) = (x-y, -3x-5y)$$

- (c) Is there a linear map
- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$
- for which
- $T(2, 2) = (8, -6)$
- and
- $T(5, 5) = (3, -2)$
- ? (Hint: Notice that
- $(5, 5) = 5/2(2, 2)$
- and explain why this causes a problem)

No. Notice that $(2, 2)$ and $(5, 5)$ are lin. dep. Indeed, $(5, 5) = 5/2(2, 2)$. (*)Supp. that $\exists T$ linear with the mentioned properties.

$$\text{Then, } T(5, 5) \stackrel{(*)}{=} T(5/2(2, 2)) \stackrel{\text{linear}}{=} 5/2 T(2, 2)$$

$$\text{But this is a } \downarrow \text{ bec. } \left. \begin{aligned} T(5, 5) &= (3, -2) \\ 5/2 T(2, 2) &= 5/2(8, -6) = (20, -15) \end{aligned} \right\} \neq$$

3. Consider the linear map defined by:

(20 pts)

$$T: M_{2 \times 2}(R) \rightarrow M_{2 \times 2}(R)$$

$$T(A) = A - A^t$$

Show that: (a) $\text{Ker}(T) = \text{Symm}(R)$

(b) $\text{Im}(T) = \text{AntiSymm}(R)$

(a) Let $A \in \text{Ker}(T) \Leftrightarrow T(A) = 0 \Leftrightarrow A - A^t = 0 \Leftrightarrow A = A^t \Leftrightarrow A \in \text{Symm}(R)$
 ie, $\text{Ker}(T) = \text{Symm}(R)$. //

(b) Let $B \in \text{Im}(T) \Rightarrow B = T(A)$, some $A \in M_{2 \times 2}(R)$.
 But $B^t = T(A)^t = (A - A^t)^t = A^t - A = -(A - A^t) = -T(A) = -B$
 ie, $B^t = -B$
 ie, $B \in \text{AntiSymm}(R)$
 ie, $\text{Im}(T) \subset \text{AntiSymm}(R)$ (1)

Conversely, let $B \in \text{AntiSymm}(R) \Rightarrow B^t = -B$ (*)
 But then, for $B \exists A = \frac{1}{2}B$ s.t. $B = \frac{1}{2}B + \frac{1}{2}B \stackrel{(*)}{=} \frac{1}{2}B - (\frac{1}{2}B)^t = T(\frac{1}{2}B) = T(A)$.
 ie, $B \in \text{Im}(T)$.
 ie $\text{AntiSymm}(R) \subset \text{Im}(T)$ (2)

4. Let L be an invertible $n \times n$ matrix and consider the linear map defined by:

$$T: M_{n \times n}(R) \rightarrow M_{n \times n}(R)$$

$$T(A) = LA$$

By (1) and (2) we have $\text{Im}(T) = \text{AntiSymm}(R)$. //

Show that T is one-to-one and onto.

(20 pts)

one-to-one: Let $T(A) = T(B) \Rightarrow LA = LB \stackrel{L \text{ invertible}}{\Rightarrow} L^{-1}(LA) = L^{-1}(LB)$
 $\Rightarrow (L^{-1}L)A = (L^{-1}L)B$
 $\Rightarrow IA = IB$
 $\Rightarrow A = B$, ie T one-to-one //

(or) Show that $\text{Ker}(T) = \{0\}$.
 Indeed, let $A \in \text{Ker}(T) \Rightarrow T(A) = 0$
 $\Rightarrow LA = 0$
 $\stackrel{L \text{ inv.}}{\Rightarrow} L^{-1}(LA) = 0$
 $\Rightarrow A = 0$
 $\Rightarrow \text{Ker}(T) = \{0\}$ //

onto: For any $B \in M_{n \times n}(R) \exists A = L^{-1}B \in M_{n \times n}(R)$ s.t. $B = T(A)$.
 Indeed, $T(A) = T(L^{-1}B) = L(L^{-1}B) = (LL^{-1})B = B$.
 ie, T onto. //

5. Consider the linear map ("differentiation") defined by:

(10 pts)

$$D: P_n[x] \rightarrow P_{n-1}[x]$$

$$D(p(x)) = p'(x)$$

Find $\text{Ker}(D)$ and use that to conclude that D is onto.

(Hint: Use the Dimension Formula for the 2nd half of the problem)

Let $p(x) \in \text{Ker}(D) \Rightarrow D(p(x)) = 0 \Rightarrow p'(x) = 0 \Rightarrow p(x) = C$, $C \in R$
 ie, $\text{Ker}(D) = \{p(x) \in P_n[x] \mid p(x) = C\} = \{C \mid C \in R\}$.

That means, $\dim(\text{Ker}(D)) = 1$.

From the Dimension Formula, we have: $\dim P_n[x] = \dim(\text{Ker}(D)) + \dim(\text{Im}(D))$

$$\Rightarrow n+1 = 1 + \dim(\text{Im}(D))$$

$$\text{ie, } \dim(\text{Im}(D)) = \dim P_{n-1}[x]$$

$$\Rightarrow \dim(\text{Im}(D)) = n$$

$\Rightarrow \text{Im}(D) = P_{n-1}[x]$, ie D onto. //