Key

TEST 3

(Math 250 A)

1. (a) Define what it means for a linear map $T: V \to W$ to be an *isomorphism.* (30 pts)

A linear map $T: V \to W$ is called an isomorphism iff T is one-to-one and onto. //

(b) Suppose that $\dim V = \dim W$. Then show that a linear map $T: V \to W$ is an isomorphism if and only if it is one-to-one or onto.
(*Hint*: Use the Dimension Formula)

By the Dimension Formula we have: $\dim V = \dim(\ker T) + \dim(\mathrm{Im}(T))$ (1)

Now, if T is onto $\Rightarrow \dim \mathrm{Im}(T) = \dim W \overset{(1)}{\Rightarrow} \dim V = \dim\ker T + \dim W \overset{\dim V = \dim W}{\Rightarrow} \dim\ker T = 0$ ie T 1-1.

if T is 1-1 $\Rightarrow \dim\ker T = 0 \overset{(1)}{\Rightarrow} \dim V = \dim(\mathrm{Im}(T)) \overset{\dim V = \dim W}{\Rightarrow} \dim W = \dim(\mathrm{Im}(T))$ ie T onto. //

(c) Show that the following linear map is an isomorphism:

$$T: M_{2\times 2}(R) \to R^4$$

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a+b+c,\, b+c,\, c,\, d)$$

(*Hint*: Use part (b) above)

Since $\dim(M_{2\times2}(\mathbb{R})) = 4 = \dim \mathbb{R}^4$, we only need to show that T is 1-1 to be an isomo.

Now, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \ker T$ iff $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (0,0,0,0) \Leftrightarrow (a+b+c, b+c, c, d) = (0,0,0,0)$

$\Leftrightarrow \begin{cases} a+b+c=0 \\ b+c=0 \\ c=0 \\ d=0 \end{cases} \Rightarrow \begin{matrix} a=0 \\ b=0 \\ c=0 \\ d=0 \end{matrix} \downarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$

ie, $\ker T = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$. So, T is 1-1 //

2. (a) **True or False**: If the linear map $T: V \to W$ is an isomorphism, then T sends a basis of V to a basis of W. (30 pts)

True //

(b) Let $T: R^2 \to R^2$ be linear, defined by: $T(x, y) = (x,\, y - x)$. Explain why T has an inverse T^{-1}. Find a formula for T^{-1}.
(*Hint*: It will be useful to see where T sends the natural basis of R^2)

T has an inverse bec. T is an isomo.

Indeed, T carries the basis $\{(1,0),(0,1)\}$ to the basis $\{(1,-1),(0,1)\}$, hence T is an isomo.

Now, T^{-1} is determined by: $T^{-1}(1,-1) = (1,0)$ and $T^{-1}(0,1) = (0,1)$.

Since $(x,y) = x(1,-1) + (x+y)(0,1)$, we have:

$$\begin{aligned} T^{-1}(x,y) &= T^{-1}(x(1,-1)) + T^{-1}((x+y)(0,1)) \\ &= x\,T^{-1}(1,-1) + (x+y)T^{-1}(0,1) \\ &= x(1,0) + (x+y)(0,1) \longrightarrow T^{-1}(x,y) = (x, x+y) \end{aligned}$$ //

(c) (i) Does the linear map $T: R^4 \to R^4$, defined by:

$$T(x,y,z,t) = (x+y,\ y+z,\ z+t,\ t+x)$$

have an inverse?
(*Hint*: Look at the Ker(T))

No, bec. T is not 1-1. Indeed, $(1,-1,1,-1) \neq (0,0,0,0) \in \text{Ker}\,T$
$\downarrow$
ie $\text{Ker}\,T \neq \{0\}$.

(ii) For the linear map $T: R^2 \to R^2$, defined by $T(x,y) = (ky - x,\ y)$, $k \in R$, show that $T^{-1} = T$.
(*Hint*: It would be easier if you computed T^2)

$T^2(x,y) = T(T(x,y)) = T(ky-x, y) = (ky-(ky-x), y) = (x,y)$
ie, $T^2 = 1_{\mathbb{R}^2} \Rightarrow TT = 1_{\mathbb{R}^2} \Rightarrow T = T^{-1}$

3. Find the matrix M_T of $T: R^2 \to R^4$, defined by $T(x,y) = (2x - y, 3x + 2y, 4y, x)$, relative to the bases B = {(1,1), (1,0)} and D = "natural basis of R^4". Use the matrix to compute $T(\vec{u})$, where $\vec{u} = (1,2)$. (10 pts)
(*Hint*: Make sure that $\vec{u}$ is expressed with respect to the appropriate basis before you pass it through the matrix M_T. The $\vec{u}$ given above is with respect to the natural basis of R^2)

$T(1,1) = (2-1, 3+2, 4, 1) = (1,5,4,1) = 1(1,0,0,0) + 5(0,1,0,0) + 4(0,0,1,0) + 1(0,0,0,1)$

$T(1,0) = (2-0, 3+0, 0, 1) = (2,3,0,1) = 2(1,0,0,0) + 3(0,1,0,0) + 0(0,0,1,0) + 1(0,0,0,1)$

So, $M_T = \begin{pmatrix} 1 & 2 \\ 5 & 3 \\ 4 & 0 \\ 1 & 1 \end{pmatrix}$.

Now, $\vec{u}$ relative to B, becomes $\vec{u} = 2(1,1) + (-1)(1,0) = (2,-1)$ $\left(= \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$

So, $M_T \vec{u} = \begin{pmatrix} 1 & 2 \\ 5 & 3 \\ 4 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 8 \\ 1 \end{pmatrix}$ $(= (0,7,8,1))$

(Notice that $M_T \vec{u}$ ($\vec{u}$ wrt B) $= T(\vec{u})$ (wrt D))

4. Let $T: R^2 \to R^2$ be linear, defined by $T(x,y) = (-x, y)$, and consider the bases B = "natural basis of R^2" and D = {(1,1), (-1,0)}. Find: (30 pts)

(a) The matrix M_T of T relative to the basis B.

$T(1,0) = (-1,0) = (-1)(1,0) + 0(0,1)$
$T(0,1) = (0,1) = 0(1,0) + 1(0,1)$
$\longrightarrow M_T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) The transition matrix P from basis B to basis D.

$(1,0) = 0(1,1) + (-1)(-1,0)$
$(0,1) = 1(1,1) + 1(-1,0)$
$\longrightarrow P = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ $\left(\Rightarrow P^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\right)$

(c) Use P above to find the matrix N_T of T relative to the basis D.

$N_T = P M_T P^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$