

TEST 3
(Math 250 B)

1. (a) Define what it means for a linear map $T: V \rightarrow W$ to be an *isomorphism*. (30 pts)

A linear map $T: V \rightarrow W$ is called an isomorphism iff T is one-to-one and onto.

- (b) Suppose that $\dim V = \dim W$. Then show that a linear map $T: V \rightarrow W$ is an isomorphism if and only if it is one-to-one or onto.
(Hint: Use the Dimension Formula)

By the Dimension Formula we have:

$$\dim V = \dim(\text{Ker } T) + \dim(\text{Im}(T))$$

Now, if T is onto $\Rightarrow \dim(\text{Im}(T)) = \dim W \Rightarrow \dim V = \dim(\text{Ker } T) + \dim W \Rightarrow \dim(\text{Ker } T) = \dim V - \dim W$
 if T is 1-1 $\Rightarrow \dim(\text{Ker } T) = 0 \Rightarrow \dim V = \dim(\text{Im}(T)) \Rightarrow \dim W = \dim(\text{Im } T)$

ie T 1-1.
 $\dim(\text{Ker } T) = 0$
 $\dim V = \dim W$
 $\dim W = \dim(\text{Im } T)$
 T onto.

- (c) Show that the following linear map is an isomorphism:

$$T: M_{2 \times 2}(R) \rightarrow R^4$$

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a+b, d, c, a-b)$$

(Hint: Use part (b) above)

Since $\dim(M_{2 \times 2}(R)) = 4 = \dim R^4$, we only need to show that T is 1-1 to be an isom.

Now, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Ker } T$ iff $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (0, 0, 0, 0) \Leftrightarrow (a+b, d, c, a-b) = (0, 0, 0, 0)$
 $\Leftrightarrow \begin{cases} a+b=0 \\ d=0 \\ c=0 \\ a-b=0 \end{cases} \Leftrightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \\ d=0 \end{cases}$
 ie, $\text{Ker } T = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. So T is 1-1.

2. (a) True or False: If the linear map $T: V \rightarrow W$ is an isomorphism, then T is invertible. (30 pts)

True.

\downarrow
 $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

- (b) Let $T: R^2 \rightarrow R^2$ be linear, defined by: $T(x, y) = (x - y, x)$. Explain why T has an inverse T^{-1} . Find a formula for T^{-1} .

(Hint: It will be useful to see where T sends the natural basis of R^2)

T has an inverse bec. T is an isom.

Indeed, T carries the basis $\{(1, 0), (0, 1)\}$ to the basis $\{(1, 1), (-1, 0)\}$, hence T is an isom.

Now, T^{-1} is determined by: $T^{-1}(1, 1) = (1, 0)$ and $T^{-1}(-1, 0) = (0, 1)$.

Since $(x, y) = \gamma(1, 1) + (\gamma - x)(-1, 0)$, we have:

$$\begin{aligned} T^{-1}(x, y) &= T^{-1}(\gamma(1, 1) + (\gamma - x)(-1, 0)) \\ &= \gamma T^{-1}(1, 1) + (\gamma - x) T^{-1}(-1, 0) \\ &= \gamma(1, 0) + (\gamma - x)(0, 1) \longrightarrow T^{-1}(x, y) = (\gamma, \gamma - x) \end{aligned}$$

(c) (i) Does the linear map $T: R^4 \rightarrow R^4$, defined by:

$$T(x, y, z, t) = (x + y, y + z, z + t, t + x)$$

have an inverse?

(Hint: Look at the $\text{Ker}(T)$)

No, bec. T is not 1-1. Indeed, $(1, -1, 1, -1) \neq (0, 0, 0, 0) \in \text{Ker } T$.
ie $\text{Ker } T \neq \{0\}$

(ii) For the linear map $T: R^2 \rightarrow R^2$, defined by $T(x, y) = (ky - x, y)$, $k \in R$,

show that $T^{-1} = T$.

(Hint: It would be easier if you computed T^2)

$$T^2(x, y) = T(T(x, y)) = T(ky - x, y) = (ky - (ky - x), y) = (x, y)$$

$$\text{ie, } T^2 = 1_{R^2} \Rightarrow TT = 1_{R^2} \Rightarrow T = T^{-1}$$

3. Find the matrix M_T of $T: R^2 \rightarrow R^4$, defined by $T(x, y) = (2x - y, 3x + 2y, 4y, x)$,

relative to the bases $B = \{(1, 1), (1, 0)\}$ and $D = \text{"natural basis of } R^4\text{"}$. Use the matrix to compute $T(\vec{u})$, where $\vec{u} = (1, 2)$. (10 pts)

(Hint: Make sure that \vec{u} is expressed with respect to the appropriate basis before you pass it through the matrix M_T . The \vec{u} given above is with respect to the natural basis of R^2)

$$T(1, 1) = (2 - 1, 3 + 2, 4, 1) = (1, 5, 4, 1) = 1(1, 0, 0, 0) + 5(0, 1, 0, 0) + 4(0, 0, 1, 0) + 1(0, 0, 0, 1)$$

$$T(1, 0) = (2 - 0, 3 + 0, 0, 1) = (2, 3, 0, 1) = 2(1, 0, 0, 0) + 3(0, 1, 0, 0) + 0(0, 0, 1, 0) + 1(0, 0, 0, 1)$$

$$\text{So, } M_T = \begin{pmatrix} 1 & 2 \\ 5 & 3 \\ 4 & 0 \\ 1 & 1 \end{pmatrix}$$

Now, \vec{u} relative to B , becomes $\vec{u} = 2(1, 1) + 1(1, 0) = (2, -1) \left(= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right)$

$$\text{So, } M_T \vec{u} = \begin{pmatrix} 1 & 2 \\ 5 & 3 \\ 4 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 8 \\ 1 \end{pmatrix} \left(= (0, 7, 8, 1) \right)$$

(Notice that $M_T \vec{u}$ (\vec{u} wrt B) = $T(\vec{u})$ (wrt D))

4. Let $T: R^2 \rightarrow R^2$ be linear, defined by $T(x, y) = (x, -y)$, and consider the (30 pts) bases $B = \text{"natural basis of } R^2\text{"}$ and $D = \{(1, 1), (-1, 0)\}$. Find:

(a) The matrix M_T of T relative to the basis B .

$$T(1, 0) = (1, 0) = 1(1, 0) + 0(0, 1) \rightarrow M_T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T(0, 1) = (0, -1) = 0(1, 0) + (-1)(0, 1)$$

(b) The transition matrix P from basis B to basis D .

$$(1, 0) = 0(1, 1) + (-1)(-1, 0)$$

$$(0, 1) = 1(1, 1) + 1(-1, 0)$$

$$\rightarrow P = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \left(\Rightarrow P^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right)$$

(c) Use P above to find the matrix N_T of T relative to the basis D .

$$N_T = PM_T P^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$$