TEST 3

(Math 250 B)

1. (a) Define what it means for a linear map $T : V \to W$ to be an *isomorphism.* (30 pts)

   (b) Suppose that $\dim V = \dim W$. Then show that a linear map $T : V \to W$ is an isomorphism if and only if it is one-to-one or onto.
   (*Hint*: Use the Dimension Formula)

   (c) Show that the following linear map is an isomorphism:

$$T : M_{2\times 2}(R) \to R^4$$
$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a+b,\ d,\ c,\ a-b)$$

   (*Hint*: Use part (b) above)

2. (a) **True or False**: If the linear map $T : V \to W$ is an isomorphism, then T is invertible. (30 pts)

   (b) Let $T : R^2 \to R^2$ be linear, defined by: $T(x, y) = (x - y,\ x)$. Explain why T has an inverse T^{-1}. Find a formula for T^{-1}.
   (*Hint*: It will be useful to see where T sends the natural basis of R^2)

(c) (i) Does the linear map $T : R^4 \to R^4$, defined by:

$$T(x, y, z, t) = (x + y,\ y + z,\ z + t,\ t + x)$$

have an inverse?
(*Hint*: Look at the Ker(T))

(ii) For the linear map $T : R^2 \to R^2$, defined by $T(x, y) = (ky - x,\ y)$, $k \in R$, show that $T^{-1} = T$.
(*Hint*: It would be easier if you computed T^2)

3. Find the matrix M_T of $T : R^2 \to R^4$, defined by $T(x, y) = (2x - y, 3x + 2y, 4y, x)$, relative to the bases B = {(1,1), (1,0)} and D = "natural basis of R^4". Use the matrix to compute $T(\vec{u})$, where $\vec{u}$ = (1,2). (10 pts)
(*Hint*: Make sure that $\vec{u}$ is expressed with respect to the appropriate basis before you pass it through the matrix M_T. The $\vec{u}$ given above is with respect to the natural basis of R^2)

4. Let $T : R^2 \to R^2$ be linear, defined by $T(x, y) = (x, -y)$, and consider the bases B = "natural basis of R^2" and D = {(1,1), (-1,0)}. Find: (30 pts)

(a) The matrix M_T of T relative to the basis B.

(b) The transition matrix P from basis B to basis D.

(c) Use P above to find the matrix N_T of T relative to the basis D.