Test 4 Key

① $\det(A - xI) = 0 \Rightarrow \det\left[\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right] = 0$

$$\Rightarrow \det\begin{pmatrix} 3-x & 6 \\ 1 & 2-x \end{pmatrix} = 0$$

$$\Rightarrow (3-x)(2-x) - 6 = 0$$

$$\Rightarrow x^2 - 5x = 0 \qquad \text{(char. eqn)}$$

Now, from the C-H Thm, we have : $A^2 - 5A = 0$

$$\Rightarrow A^2 = 5A \qquad (*)$$

$$\Rightarrow A^2 = 5\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$$

ie, $A^2 = \begin{pmatrix} 15 & 30 \\ 5 & 10 \end{pmatrix}$

From $(*)$, we get $A^3 = 5A^2$ (mult. $(*)$ by A)

$$\Rightarrow A^3 = 5\begin{pmatrix} 15 & 30 \\ 5 & 10 \end{pmatrix}$$

$$\Rightarrow A^3 = \begin{pmatrix} 75 & 150 \\ 25 & 50 \end{pmatrix} \qquad (**)$$

From $(**)$, we get $A^4 = 5A^3$ (mult. $(**)$ by A)

$$\Rightarrow A^4 = 5\begin{pmatrix} 75 & 150 \\ 25 & 50 \end{pmatrix}$$

$$\Rightarrow A^4 = \begin{pmatrix} 375 & 750 \\ 125 & 250 \end{pmatrix} \qquad (***)$$

From $(***)$, we get $A^5 = 5A^4$ (mult. $(***)$ by A)

$$\Rightarrow A^5 = 5\begin{pmatrix} 375 & 750 \\ 125 & 250 \end{pmatrix}$$

$$\Rightarrow A^5 = \begin{pmatrix} 1875 & 3750 \\ 625 & 1250 \end{pmatrix}$$

ie, $A^5 = \begin{pmatrix} 1875 & 3750 \\ 625 & 1250 \end{pmatrix}$

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(2) (a) $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$

$\det(A - xI) = 0 \Rightarrow \det\left[\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} - \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}\right] = 0$

$\Rightarrow \det\begin{pmatrix} 1-x & 0 & -2 \\ 0 & -x & 0 \\ -2 & 0 & 4-x \end{pmatrix} = 0$

$\Rightarrow -x^3 + 5x^2 = 0$ (char. eqn)

Now, $-x^3 + 5x^2 = 0 \Rightarrow x^2(-x+5) = 0 \Rightarrow \begin{cases} x_1 = 0 & \text{(repeated root)} \\ x_2 = 5 \end{cases}$

So, alg. mult. of $0 = 2$ (= mult. of the root)
alg. mult. of $5 = 1$ (= mult. of the root)

The corresponding eigenspaces E_0 and E_5 are computed by solving the following systems of eqn's :

For $x_1 = 0 \longrightarrow \left\{\begin{matrix} x - 2z = 0 \\ -2x + 4z = 0 \end{matrix}\right\} \longrightarrow z = \frac{1}{2}x \Rightarrow E_0 = \{(x, y, \frac{x}{2}) \mid x, y \in \mathbb{R}\}$
$= \text{span}\{(0,1,0), (2,0,1)\}$

For $x_2 = 5 \longrightarrow \left\{\begin{matrix} -4x - 2z = 0 \\ -5y = 0 \\ -2x - z = 0 \end{matrix}\right\} \longrightarrow \begin{matrix} z = -2x \\ y = 0 \end{matrix} \Rightarrow E_5 = \{(x, 0, -2x) \mid x \in \mathbb{R}\}$
$= \text{span}\{(1, 0, -2)\}$

So, geom. mult. of $0 = 2$ ($= \dim E_0$)
geom. mult. of $5 = 1$ ($= \dim E_5$)

Since, alg. mult. of $0 = 2 =$ geom. mult. of 0.
alg. mult. of $5 = 1 =$ geom. mult. of 5

$\Rightarrow$ A is diag/ble.

It is similar to $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. //

(b) $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$

$\det(A - xI) = 0 \Rightarrow \det\left[\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} - \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}\right] = 0$

$$\Rightarrow \det\begin{pmatrix} 2-x & 1 & 0 \\ 0 & 1-x & -1 \\ 0 & 2 & 4-x \end{pmatrix} = 0$$

$$\Rightarrow -x^3 + 7x^2 - 16x + 12 = 0 \qquad (\text{char. eqn})$$

Now, $-x^3 + 7x^2 - 16x + 12 = 0 \Rightarrow -(x-2)^2(x-3) = 0$

$$\Rightarrow \left< \begin{array}{l} x_1 = 2 \quad (\text{repeated root}) \\ x_2 = 3 \end{array} \right.$$

So, alg. mult. of $2 = 2$
alg. mult. of $3 = 1$

The E_2, E_3 are computed as follows:

For $x_1 = 2 \longrightarrow \left\{ \begin{array}{l} y = 0 \\ -y - z = 0 \\ 2y + 2z = 0 \end{array} \right\} \rightarrow y = 0,\ z = 0 \quad \Rightarrow E_2 = \{(x,0,0) \mid x \in \mathbb{R}\}$
$= \text{span}\{(1,0,0)\}$

For $x_2 = 3 \longrightarrow \left[\begin{array}{l} -x + y = 0 \\ -2y - z = 0 \\ 2y + z = 0 \end{array} \right] \longrightarrow \begin{array}{l} y = x \\ z = -2x \end{array} \quad \Rightarrow E_3 = \{(x, x, -2x) \mid x \in \mathbb{R}\}$
$= \text{span}\{(1,1,-2)\}$

So, geom. mult. of $2 = 1$
geom mult. of $3 = 1$

Since alg. mult. of $2 = 2 \neq 1 =$ geom. mult. of 2

$\Rightarrow$ A is <u>not</u> diag/ble. //

③ $\|A+B\|^2 = \langle A+B, A+B \rangle$

$$= \langle A, A \rangle + \langle A, B \rangle + \langle B, A \rangle + \langle B, B \rangle$$

$$= \|A\|^2 + 2\langle A, B \rangle + \|B\|^2$$

$$\leq \|A\|^2 + 2|\langle A, B \rangle| + \|B\|^2 \qquad (\text{bcc. } x \leq |x|)$$

$$\overset{C\text{-}S}{\leq} \|A\|^2 + 2\|A\|\|B\| + \|B\|^2 \qquad (\text{by Cauchy-Schwarz Ineq.})$$

$$= (\|A\| + \|B\|)^2 \quad \xrightarrow{\sqrt{\ }} \quad \|A+B\| \leq \|A\| + \|B\|$$

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(4) (a) $\delta_k(a_i) = \dfrac{\prod_{k\neq i}(a_i - a_i)}{\prod_{k\neq i}(a_k - a_i)} = \begin{cases} i\neq k: & \dfrac{0}{\prod_{k\neq i}(a_k-a_i)} = 0 \\ i=k: & \dfrac{\prod_{k\neq i}(\cancel{a_i-a_i})}{\prod_{k\neq i}(\cancel{a_i-a_i})} = 1 \end{cases}$

(b) $\langle \delta_i(x), \delta_j(x)\rangle_{i\neq j} = \sum_{k=0}^{n} \delta_i(a_k)\delta_j(a_k) = 0.0 + \cdots + 0.0 + \underset{\substack{\uparrow \\ k=i}}{1.0} + 0.0 + \cdots + 0 = 0$

$\langle \delta_i(x), \delta_j(x)\rangle_{i=j} = \sum_{k=0}^{n} \delta_i(a_k)\delta_i(a_k) = 0.0 + \cdots + 0.0 + \underset{\substack{\uparrow \\ k=i}}{1.1} + 0.0 + \cdots + 0 = 1$

So, $\delta_i(x) \perp \delta_j(x)$, $\forall i\neq j$, and $\|\delta_i(x)\| = 1$, $\forall i$.

ie, $\{\delta_i(x)\}_{i=0,1,\dots,n}$ is an orthonormal basis.

(c) $\langle p(x), \delta_k(x)\rangle = p(a_0)\delta_k(a_0) + p(a_1)\delta_k(a_1) + \cdots + p(a_k)\delta_k(a_k) + \cdots + p(a_n)\delta_k(a_n)$

$= p(a_0)\cdot 0 + p(a_1)\cdot 0 + \cdots + p(a_k)\cdot 1 + \cdots + p(a_n)\cdot 0$

$= p(a_k)$

(d) Since $\{\delta_i(x)\}_{i=0,1,\dots,n}$ is an orthonormal basis, from the Expansion Theorem, we have:

$$p(x) = \frac{\langle p(x), \delta_0(x)\rangle}{\|\delta_0(x)\|^2}\delta_0(x) + \cdots + \frac{\langle p(x), \delta_n(x)\rangle}{\|\delta_n(x)\|^2}\delta_n(x)$$

$$\overset{(c)}{\underset{(b)}{=}} \frac{p(a_0)}{1}\delta_0(x) + \cdots + \frac{p(a_n)}{1}\delta_n(x)$$

$$= p(a_0)\delta_0(x) + \cdots + p(a_n)\delta_n(x)$$

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(5) Basis B is <u>not</u> orthonormal, bec. is neither orthogonal nor its elmts have length 1. For example:

$\langle A, B\rangle = Tr(AB^t) = 1.0 + 1.1 + 0.0 + 0.1 = 1 \neq 0$, ie $A \not\perp B$

$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{Tr(AA^t)} = \sqrt{1^2+1^2+0^2+0^2} = \sqrt{2} \neq 1$

Now, using G-S, we get :

$$A' = A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B' = B - \frac{\langle B, A' \rangle}{\|A'\|^2} A' = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$$

$$C' = C - \frac{\langle C, A' \rangle}{\|A'\|^2} A' - \frac{\langle C, B' \rangle}{\|B'\|^2} B' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 & -2/3 \\ 1 & -1/3 \end{pmatrix}$$

$$D' = D - \frac{\langle D, A' \rangle}{\|A'\|^2} A' - \frac{\langle D, B' \rangle}{\|B'\|^2} B' - \frac{\langle D, C' \rangle}{\|C'\|^2} C'$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -1/2 & 1/2 \\ 0 & 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 2/3 & -2/3 \\ 1 & -1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 5/9 & -5/9 \\ -1/6 & 13/18 \end{pmatrix}$$

The basis $\mathcal{B}' = \{A', B', C', D'\}$ is orthogonal.

Now, $$A'' = \frac{A'}{\|A'\|} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{pmatrix}$$

$$B'' = \frac{B'}{\|B'\|} = \begin{pmatrix} 2/3\sqrt{3} & -2/3\sqrt{3} \\ 1/\sqrt{3} & -1/3\sqrt{3} \end{pmatrix}$$

$$C'' = \frac{C'}{\|C'\|} = \begin{pmatrix} 2/3\sqrt{2} & -2/3\sqrt{2} \\ 1/\sqrt{2} & -1/3\sqrt{2} \end{pmatrix}$$

$$D'' = \frac{D'}{\|D'\|} = \begin{pmatrix} 10/\sqrt{378} & -10/\sqrt{378} \\ -3/\sqrt{378} & 13/\sqrt{378} \end{pmatrix}$$

So, the basis $\mathcal{B}'' = \{A'', B'', C'', D''\}$ is orthonormal.

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