TEST 4

(Math 250 A, B)

(Take Home)

1. The *Caley-Hamilton Theorem* provides a method for computing powers of a matrix A: (20 pts)

   e.g. Let A be a 2x2 matrix with characteristic equation $x^2 + ax + b = 0$.
   Then, by the Caley-Hamilton Theorem, we have:
   $$A^2 + aA + bI = 0$$
   which gives:
   $$A^2 = -aA - bI\ .$$
   In other words, we just computed A^2. Multiplying both sides by A, we get:
   $$A^3 = -aA^2 - bA$$
   So, we just computed A^3, because the last equation is given in terms of the known quantities A^2 and A. Continuing in this way, we can compute A^n, for some n, simply by expressing A^n in terms of lower powers of A .

   Use the above procedure to compute A^5 for $A = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$.

2. Consider the following Definition and the Theorem after that: (20 pts)

   Def.: Let A be a nxn matrix and λ an eigenvalue of A. The **geometric multiplicity** of λ is the dimension of the eigenspace E_λ. The **algebraic multiplicity** of λ is the multiplicity of the root λ of the characteristic polynomial $p_A(x)$.

   Thm: A matrix A is diagonalizable if and only if : (1) $p_A(x)$ is a product of linear factors.
   (2) For each λ the two multiplicities are equal.

   Do not prove the Theorem. Instead, used it to check whether the following matrices are diagonalizable:

   (a) $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$ (b) $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$

   In case one of them is diagonalizable, write down the diagonal matrix D with which A is similar to.

3. Prove the *Triangle Inequality*. In other words, show that for every A and B in V (where V is an inner product space), we have: (20 pts)

$$\| A+B \| \leq \| A \| + \| B \| \ .$$

4. Consider the distinct numbers $a_0, a_1, ..., a_n$ and $p(x), q(x) \in P_n[x]$. Define: (20 pts)

$$< p(x), q(x) > = p(a_0)q(a_0) + p(a_1)q(a_1) + ... + p(a_n)q(a_n)$$

Clearly, the above is an inner product in $P_n[x]$. Now, define the **Lagrange Polynomials** $\delta_0(x), \delta_1(x), ..., \delta_n(x)$ relative to $a_0, a_1, ..., a_n$ as follows:

$$\delta_k(x) = \frac{\prod_{i \neq k}(x - a_i)}{\prod_{i \neq k}(a_k - a_i)} \quad , \quad \text{with } k = 0,1,...,n \quad ,$$

where $\prod_{i \neq k}(x - a_i)$ denotes the product of all the terms $(x - a_0), (x - a_1), ..., (x - a_n)$ except the *k*th term. Now, show the following:

(a) $\delta_k(a_i) = \begin{cases} 0 \ , \ i \neq k \\ 1 \ , \ i = k \end{cases}$.

(b) $\{\delta_0(x), \delta_1(x), ..., \delta_n(x)\}$ is an orthonormal basis.

(c) $< p(x), \delta_k(x) > = p(a_k)$.

(d) $p(x) = p(a_0)\delta_0(x) + p(a_1)\delta_1(x) + ... + p(a_n)\delta_n(x) \ , \ \forall \ p(x) \in P_n[x]$.

(*Note*: For part (d) use the Expansion Theorem. Part (d) is called the **Lagrange Interpolation Expansion** of $p(x)$ and is very important in numerical integration)

5. Given the basis B = { $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ }, check whether it is an orthonormal basis. If your answer is **no**, then use the *Gram-Schmidt Orthogonalization Process* to construct an orthonormal basis out of the one above. (20 pts)