Problem Set 1

(Operations, Groups, Rings)

1. Let $A = \{a \in Z \mid a = x^2 + y^2, x, y \in Z\}$. Show that A is closed under multiplication.

2. Let $M = \{A \in M_{2\times2}(R) \mid A = aI + bX,\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, a, b \in R\}$. Show that A is closed under multiplication.

3. Consider the set $A = \{2, 3\}$ and its power-set $P(A)$. Show that the operation "$\cup$" is an inner operation on $P(A)$.

4. Consider the operation $*$ on $R^* = R\text{-}\{0\}$, defined by $a * b = \dfrac{1}{a} + \dfrac{1}{b}$. Examine whether R^* is closed under $*$.

5. Consider the operation $\oplus$ on Z, defined by $a \oplus b = a + b - 2$. Find:
(a) The *identity* element wrt $\oplus$.
(b) The *inverse* of 7 wrt $\oplus$.
(c) Is $\oplus$ associative?
(d) Evaluate $K = \dfrac{1 \oplus 4 - 2}{3(2 + 5 \oplus 1)}$

6. Consider the operation $\bullet$ on Q, defined by $a \bullet b = a + b - ab$. Show that:
(a) Q is an Abelian semigroup wrt $\bullet$.
(b) Is Q a monoid?
(c) Is Q a group?

7. Show that the set $SL_2(R) = \{A \in M_{2\times2}(R) \mid \det(A) = 1\}$ is a group (***Special Linear Group***).

8. Show that every group G with three elements is cyclic.

9. (a) Show that the set $K = \{e, a, b, c\}$ with the following multiplication table:

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

is a group (***Klein four-group***).

(b) Is the set $L = \{e, a, b, c\}$ with the following multiplication table:

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	c	e	a
b	b	a	c	e
c	c	b	a	e

a group?

10. (a) Show that the set $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, where:

$$\pm 1 = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \pm i = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}, \pm j = \begin{pmatrix} 0 & \pm i \\ \pm i & \pm 1 \end{pmatrix}, \pm k = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}$$

is a group (***Quaternion group***).
(b) Is Q Abelian?
(c) What is the *center* Z(Q) of Q?
(d) Construct the multiplication table of the group Q.

11. (a) Show that the set $R = \{a + bi \mid a, b \in Z\} \subset C$, where C is the set of complex numbers, is an Abelian ring. Is it a field? (***Gaussian Integers***).
(b) Show that C is a field.

12. (a) Show that the set $M_{2\times 2}(R)$ is a non-Abelian ring.
(b) Show that the set $D = \{A \in M_{2\times 2}(R) \mid A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\ a, b \in R\}$ is an Abelian ring.

Is D a field? Does it have *zero-divisors*?

(c) Show that the set $F = \{A \in M_{2\times 2}(R) \mid A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},\ a \in R\}$ is a field.

13. If R is a ring with at least two elements, then $0 \neq 1$.
(*Note*: 0 and 1 are the additive and multiplicative identities respectively. Notice that if $0 = 1$, then R has one element and that is 0)