MAT 351
(Spring 08)

Problem Set 3
(Complex Numbers)

1. Write the following complex numbers in the form $a+bi$:

(i) $i^5(1-i)^2$ (ii) $\dfrac{1+2i}{3-7i}$ (iii) $\dfrac{1-3i}{2+i}+\dfrac{5+2i}{2i}$

2. (a) Show that Re(iz) = - Im(z).

(b) Let $z \in$ C be such that Im(z) > 0. Show that Im(1/z) < 0.

3. (a) Investigate the values obtained by raising i in different powers $n \in$ N.
(*Hint*: Consider viewing any n as $n = 4k + r$, for some r = 0,1,2,3)

(b) Use your values in (a) to compute $2i^5+5i^2-i^{10}+\dfrac{3}{i^7}$.

4. (a) Let $z=2+3i$. Plot the numbers $z, -z, \bar{z}, -\bar{z}, 1/z$ on the complex plane.

(b) Show that the numbers $z_1=1+i$, $z_2=1+4i$, $z_3=3+i$ consist the vertices of a right-angle triangle.

(c) Describe the set of points in the complex plane that satisfy the following:
(i) Im(z) = -3 (ii) $|z-i| = 9$ (iii) $|z-i| = |z+1|$ (vi) $|z-1| + |z+1| = 5$ (v) $|z+i| < 3$

5. (a) Show that $|z-1|=|\bar{z}-1|$.

(b) Show that if $|z|=1$, where $z \neq 1$, then Re[1/(1-z)] = 1/2.

(c) Examine whether the set $K = \{ z \in C \mid |z| \in [0,1]\}$ forms a *group* under multiplication.

6. (a) Show that z_1, z_2 are *parallel* iff Im($z_1\bar{z}_2$) = 0.

(b) Show that any point z on the line through points z_1, z_2 is of the form
$z = z_1 + c(z_2 - z_1)$, where $c \in$ R.

(c) Show that $|z^k| = |z|^k$, $\forall\, k \in$ Z.

(d) Evaluate: (i) $\left|\dfrac{1-2i}{2+3i}\right|$ (ii) $|(1+i)\overline{(2-5i)}2i|$

7. (a) Show that $|z_1 - z_2| \leq |z_1| + |z_2|$.

(b) Show that $||z_1| - |z_2|| \leq |z_1 - z_2|$.

(c) Show that $|z_1 + z_2| = |z_1| + |z_2|$ iff z_1, z_2 have the same argument.
(*Hint*: Show that geometrically)

8. (a) Write the following complex numbers in *polar form*:

(i) 3 (ii) $-2i$ (iii) $1-\sqrt{3}i$ (iv) $\dfrac{1-2i}{2+i}$

(b) **True or False**: (i) Arg($z_1 \cdot z_2$) = Arg(z_1) + Arg(z_2) (ii) Arg($\bar{z}$) = -Arg(z)

9. (a) Use De Moivre's formula to show that $\left(\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)^{26}=3^{26}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$.

(b) Find: (i) $\sqrt[3]{1}$ (ii) $\sqrt[6]{i}$ (iii) $\sqrt{1+i}$

(c) Show that the 3rd-roots of 1 you found in (b)(i) above form a *cyclic group* C. What is $|C|$?

(d) Use **De Moivre's Formula** to show that $\sin(3\theta)=3\cos^2\theta\sin\theta-\sin^3\theta$.

(e) Solve the equation $z^2-z+i=0$.

10. (a) Write each of the following functions in the form $w=u(y,y)+v(x,y)i$:

(i) $f(z)=2z^2-3z+2i-1$ (ii) $f(z)=\frac{2-i}{z}$ (iii) $f(z)=\frac{z-i}{z^2+1}$

(b) Find the *domain* of each of the function in (a) above.

11. (a) Show that $\lim_{n\to\infty} z_z=0$ iff $\lim_{n\to\infty}|z_z|=0$.

(b) Show that if $|z_z|<1$, then $z_z\to 0$ as $n\to\infty$.

(c) Do the following sequences converge? If yes, where?

(i) $z_z=\frac{i+1}{2n}$ (ii) $z_z=2i(-1)^n$ (iii) $z_z=\frac{n(3i-1)}{2ni}$

(d) Show that the $\lim_{z\to -i} z^2=-1$, using the definition.

(e) Find the following limits:

(i) $\lim_{z\to i+1}(z^2-2i)$ (ii) $\lim_{z\to i}\frac{z-i}{z^2-1}$ (iii) $\lim_{z\to 4i}\frac{z-4}{z^2+16}$

12. (a) Show that $f(z)=\bar{z}$ is continuous everywhere on the plane.

(b) Is $f(z)=\frac{z+i}{z(z-2i)}$ continuous on the whole plane?

13. (a) Show that if $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0.

(b) Is the converse of (a) true?

(c) Find the derivatives of the following functions:

(i) $f(z)=2z^3-3iz+i+1$ (ii) $f(z)=(3z^2-i)^{-3}$ (iii) $f(z)=\frac{z-i+2}{(z-i)^2}$

(d) Is $f(z)=\bar{z}$ differentiable anywhere on the whole plane?

(e) Is $f(z)=\frac{z^2+5i}{2z-i}$ differentiable everywhere on the whole plane?

(f) Find $\lim_{z\to -i}\frac{z^4-1}{z^{12}-1}$. (*Hint*: Use **L'Hospital's Rule**)

14. (a) Show that $f(z)=\text{Re}(z)$ is nowhere differentiable.

(b) Show that $f(z)=(x^3+x)+(y^2)i$ is differentiable only on the line $y=(3/2)x+1/2$.

(*Hint*: Use **Cauchy-Riemann Equations** for all the above)