MAT 351
(Spring 08)

Problem Set 2

(Vector Spaces, Algebras)

1. In a real vector space V, show that $0A = \mathbf{0}$, for all A in V.

2. Show that the following sets, with their natural operations of addition and scalar multiplication, are vector spaces:
(a) C (the complex numbers)
(b) $M_{2\times2}(\mathrm{R})$

3. Could the set $V = \{(x,y,z) \in \mathrm{R}^3 \mid x+y+2z=1\}$, with the usual addition and scalar multiplication of R^3, be a vector space? Why?

4. (a) The set $\mathrm{R}^+ = \{x \in \mathrm{R} \mid x \geq 0\}$, equipped with the following *addition* and *scalar multiplication*:

$$x + y = xy$$
$$\lambda \cdot x = x^\lambda$$

is a vector space. In this vector space:
(i) What is $\mathbf{0}$? (the *zero* element of V). What is $-\boldsymbol{x}$? (the *opposite* of an element $\boldsymbol{x}$)
(ii) Show that $5+2=10$ and $0 \cdot 8 = 1$?
(b) Examine whether the set of rational numbers Q is a subspace of R.

5. (a) Let $W = \{p(x) \in P_2[x] \mid p(x) = p(-x)\}$. Show that W is a subspace of $P_2[x]$.
(b) Describe, in general, the form that a polynomial $p(x)$ has in W.

6. Let $W = \{A \in M_{2\times2}(\mathrm{R}) \mid \det(A) = 1\}$. Is W a subspace of $M_{2\times2}(\mathrm{R})$? If your answer is *no*, provide a counterexample.

7. (a) Find a basis for the vector space $M_{3\times3}(\mathrm{R})$, other than the standard basis.
(b) The set of rotations in R^3 is the set defined by:

$$SO(3) = \{A \in M_{3\times3}(\mathrm{R}) \mid AA^t = I, \det(A) = 1\} \quad (\textit{\textbf{Special Orthogonal Group}})$$

It is not hard to show that $SO(3)$ is a group. Is it a subspace of $M_{3\times3}(\mathrm{R})$ though?

8. (a) Do the vectors x = (3,1,-4,6), y = (1,1,4,4), z = (1,0,-4,1), w = (-2,-2,-8,-8) of R^4 form a basis for R^4? Why?
(b) Do the first three vectors, namely x, y, z, form a 3-dimensional subspace of R^4?
(c) What is the largest dimension that a subspace W of R^4 can have? Find W.

generated by x,y,z,w

9. (a) Show that $M_{2\times 2}(\mathrm{R})$ is an associative, non-commutative, real algebra with identity, and with zero-divisors.

(b) Show that its subset $SO(2)$ is commutative though.

(c) Similarly, we know that $M_{3\times 3}(\mathrm{R})$ is also a non-commutative algebra. Is its subset $SO(3)$ commutative though?

10. The space R^3 with $\times$ (the "cross product") is a non-associative, non-commutative, real algebra without an identity, and with zero-divisors. Is it a ***Lie Algebra***?

11. The space $Symm_{2\times 2}(\mathrm{R}) = \{A \in M_{2\times 2}(\mathrm{R}) \mid A = A'\}$, with a multiplication defined by:

$$A \cdot B = \frac{AB + BA}{2}$$

becomes a non-associative, commutative, real algebra with identity, and with zero-divisors.

(a) What is the dimension of the algebra $Symm_{2\times 2}(\mathrm{R})$?

(b) Is it a ***Jordan Algebra***?

12. (a) Show that the set $B = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in M_{2\times 2}(\mathrm{R}) \mid a, b \in \mathrm{R}\}$ is a subalgebra of $M_{2\times 2}(\mathrm{R})$.

(b) What is its dimension?

(c) Find an algebra-homomorphism that will help you establish that $\mathrm{C} \cong B$.

13. (a) Show that C is an associative, commutative, ***division algebra***.

(b) C with a multiplication defined by:

$$z \cdot w = \overline{zw}$$

is also a commutative division algebra. Is it associative? Does it have an identity?