MAT 351
(Spring 08)

Problem Set 4
(Quaternions)

1. (a) Given the quaternions $q = 2 - i + 3k$, $p = 1 - j - k$, find the following:

(i) $q + p$ (ii) qp (iii) $-7p$ (iv) $\overline{q}$ (v) $\| p \|$ (vi) q^{-1}

(b) **True or False:** $\| q + p \| = \| q \| + \| p \|$.

(c) Find the angle θ between q and p.

2. (a) Let $p, q, r \in$ H. Show that:

(i) $p(qr) = (pq)r$ (ii) $p(q + r) = pq + pr$ (iii) $\overline{pq} = \overline{q}\,\overline{p}$ (iv) $\overline{p + q} = \overline{p} + \overline{q}$

(*Hint*: Use the [,] notation)

(b) We define *division* by $p / q := pq^{-1}$. **True or False:** $pq^{-1} = q^{-1} p$.

3. (a) Let $p, q \in$ Im(H). Show that $qp = -(q \cdot p) + q \times p$, where "$\cdot$" and "$\times$" are the "dot" and "cross" products in R^3, respectively.

(b) Show that $q \perp p$ iff q and p anti-commute.

(c) Let $p, q, r \in$ Im(H). Show that $q \times (p \times r) = \frac{1}{2}(qpr - prq)$.

4. Let $H_1 = \{ q \in H \mid \| q \| = 1 \}$ (quaternions of length 1) and $H^0 = \{ q \in H \mid q \neq 0 \}$ (non-zero quaternions).

(a) Show that H_1 is a *normal* subgroup of H^0.

(b) Is the *quaternion group* $Q = \{ \pm 1, \pm i, \pm j, \pm k \}$ normal in H_1 ?

(c) **True or False:** Geometrically speaking, H_1 is the *unit-hypersphere* S^3 in R^4. Explain.

5. (a) Show that all subgroups of the *quaternion group* $Q = \{ \pm 1, \pm i, \pm j, \pm k \}$ are normal in Q.

(*Note*: Non-commutative groups that all their subgroups are normal are called **Hamiltonian groups**)

(b) Show that every $a \in Q$, $a \neq \pm 1$, has *order* 4.

6. (a) Let $\mathtt{H} = \{ \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix} \in M_{2\times 2}(C) \mid a, b \in C \}$ be the subalgebra of $M_{2\times 2}(C)$. Show that $\mathtt{H} \cong H$.

(b) Show that any $A \in \mathtt{H}$ is a root of the quadratic equation $X^2 - Tr(X)X + Det(X)I = 0$, where $Tr(X) = 2\mathrm{Re}(z)$ and $Det(X) = |z|^2 + |w|^2$.

(c) Construct a matrix A in $M_{2\times 2}(H)$ such that A is invertible, but $Det(A) = 0$.

7. (a) Let q be the quaternion $q = \frac{1}{2} - \frac{\sqrt{2}}{2} i + \frac{1}{2} k$. Write q in the *trigonometric form* $q = [\cos\theta, w \sin\theta]$, where $\| w \| = 1$. Then, find log$q$.

(b) Let q be the quaternion $q = \frac{\pi}{3\sqrt{3}} i - \frac{\pi\sqrt{2}}{3\sqrt{3}} k$. Write q in the form $q = [0, \theta w]$, where $\| w \| = 1$. Then, find e^q.

(c) Let $q = j - k$. Find $\sqrt[3]{q^2}$.

(*Hint*: For (c), consider finding $q^{2/3}$)

8. (a) Let $q = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4} i + \frac{1}{2} j + \frac{\sqrt{2}}{4} k$. Then:

(i) Show that $q \in H_1$ (i.e. $q \in S^3$).

(ii) Express q in *polar* form (i.e. $q = e^{\theta w}$).

(iii) Find log$q$.

(iv) Find $q^{1/3}$

(b) Solve the equation $x^2 = i$ in H.

9. True or False: (i) $\log(qp) = \log q + \log p$ (ii) $e^q e^p = e^{q+p}$ (iii) $e^q = 1 \Leftrightarrow q = 0$

10. Define $e^q := \sum_{n=0}^{\infty} \frac{q^n}{n!}$, $q \in$ H. (quaternionic exponential function)

(a) Let $q = [s, \theta w]$, where $s, \theta \in$ R and $\| w \| = 1$. Show that:

(i) $e^{\theta w} = \cos\theta + w \sin\theta$ (ii) $e^q = e^s e^{\theta w}$ (iii) $\| e^{\theta w} \| = 1$ (iv) $\| e^q \| = e^s$

(b) **True or False:** $e^q e^p = e^{q+p} \Leftrightarrow qp = pq$.

11. (a) Let q be the quaternion $q = \frac{3}{5} i + \frac{4}{5} j$. Show that $q^2 = -1$. In other words, q is another "square root of –1", like *i*, *j* and *k*. Can you find more quaternions like *q*?

(b) Show that any quaternion q of the form $q = \frac{a}{c} i + \frac{b}{c} j$, where *a*, *b*, and *c* satisfy Pythagora's Theorem (i.e. $c^2 = a^2 + b^2$) is a square root of –1.

(c) Considering (b), how many square roots of –1 does H has, other than like *i*, *j* and *k*? If you say ∞- many, does this contradict with the fact that H is 4-dimensional?

12. (a) Construct a quaternion *q* that rotates by $45°$ about the *z*-axis.

(b) Construct a quaternion that performs 3-times the rotation in (a).

(c) What type of rotation is represented by the quaternion:

$$q = 0.865 + 0.137i - 0.137j + 0.137k$$

(d) Construct a quaternion that performs 1/8th of the rotation in (c).

(*Hint*: For (b) and (d), use the exponential formula for q^t)

13. (a) Use a non-unit quaternion *q* in $R_q(w) = qwq^{-1}$. What do you observe?

(b) Verify that *q* and –*q* represent the same rotation.

(c) Verify that $(R_q \circ R_p)(w) = R_{qp}(w)$.